

L^p estimates for fully coupled FBSDEs with jumps *

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Abstract. In this paper we study useful estimates, in particular L^p -estimates, for fully coupled forward-backward stochastic differential equations (FBSDEs) with jumps. These estimates are proved at one hand for fully coupled FBSDEs with jumps under the monotonicity assumption for arbitrary time intervals and on the other hand for such equations on small time intervals. Moreover, the well-posedness of this kind of equation is studied and regularity results are obtained.

Keyword. Fully coupled FBSDEs with jumps; L^p -estimates

1 Introduction

General nonlinear backward stochastic differential equations (BSDEs, for short) driven by a Brownian motion were introduced and studied by Pardoux, Peng in [10]. Since that pioneering paper from 1990, the theory of BSDEs has been intensively studied by a lot of researchers attracted by its various applications, namely in stochastic control (see Peng [13]), finance (see El Karoui, Peng and Quenez [3]), and the theory of partial differential equations (PDEs, for short) (see Pardoux, Peng [11], Peng [14], etc).

The study of BSDEs has led also to generalizations, among them BSDEs driven by both a Brownian motion and an independent Poisson random measure (first studied by Tang and Li [16]) but also fully coupled forward-backward stochastic differential equations (FBSDEs) governed by a Brownian motion and such FBSDEs governed by both a Brownian motion and Poisson random measure.

As concerns the fully coupled FBSDEs driven by a Brownina motion, they were intensively studied under different assumptions by different authors. While Ma and Yong [8] developed under the assumption of strict ellipticity of the diffusion coefficient of the forward equation the so-called 4-step scheme for FBSDE, Hu and Peng [4], Peng and Wu [15] studied FBSDEs under the so-called monotonicity assumption, while Pardoux and Tang [12] used a different condition. All these three conditions are of different type and not really comparable. In recent works Ma, Wu, Zhang and Zhang [9] have studied fully coupled FBSDE which involve these three types of conditions.

Fully coupled FBSDEs driven by both a Brownian motion and a Poisson random measure were studied by Wu [17], [18] under the monotonicity condition. For this he extended the arguments of [4], [15] to the case with jumps. While in [17] he obtained the existence and the uniqueness for such fully coupled FBSDEs with jumps, in Wu [18] he proved the existence and the uniqueness of the solution as well as a comparison theorem for fully coupled FBSDEs with jumps over a stochastic interval.

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The main objective of our paper is to study useful estimates, in particular L^p estimates for fully coupled FBSDEs with jumps which are not the same as L^p estimates for fully coupled FBSDEs driven only by a Brownian motion, refer to Proposition 3.2, Remark 3.4, and Theorem 3.4. These estimates, particularly challenging for the case of fully coupled FBSDEs with jumps, have been already well studied for fully coupled FBSDEs driven only by a Brownian motion. We refer the reader, in particular, to the paper [2] by Delarue. His results and estimates for fully coupled FBSDEs driven only by a Brownian motion over a sufficiently small time interval were extended by Li and Wei [6] to controlled fully coupled FBSDEs in the frame of their study of an optimal stochastic control problem with coupling between the controlled forward and the controlled backward equation, while, in particular, the diffusion coefficient of the forward equation σ depends on z . In the frame of their studies they proved some new L^p -estimates for fully coupled FBSDEs on small time interval which were crucially used for the link between the stochastic control problem and the associated system of PDEs formed by a quasi-linear Hamilton-Jacobi-Bellman (HJB, for short) equation and an algebraic equation.

Inspired by the control problems studied by [1], [5] and [6], Li, Wei [7] have investigated recently stochastic differential games defined through fully coupled FBSDEs with jumps. These studies have required specific types of non-trivial L^p -estimates for fully coupled FBSDE with jumps, which have also their own interest. They extend former results for coupled FBSDEs without jumps and are based on rather technical proofs.

In this paper, we first study L^2 -estimates (Proposition 3.1) and L^p -estimates (Proposition 3.2) for fully coupled FBSDEs with jumps under the monotonicity condition. In our proofs we use a new method, in particular in the proof of Proposition 3.2; the estimates (3.11) and (3.15) concerning the jump martingale part turn out to be crucial for other estimates in this work.

In the second part of our paper, assuming the Lipschitz coefficients with respect to z and k of the diffusion coefficient and the coefficient in the jump integral to be sufficiently small, we first prove the existence and uniqueness (Theorem 3.2) of the solution of fully coupled FBSDEs with jumps on a small time interval and also a generalized Comparison Theorem (Theorem 3.3). Then we derive the L^p -estimates (Theorem 3.4) for fully coupled FBSDEs with jumps on the small time interval. This second part provides estimates which turn out to be crucial in the study of stochastic differential games and for the study of the existence of the viscosity solution for the associated second order integral-partial differential equation of Isaacs' type over an arbitrary time interval, combined with an algebraic equation; see [7]. Of course, the results of our paper can be also applied to the study of other problems, as for instance, the optimal control problems and the stochastic maximum principle of fully coupled FBSDEs with jumps.

This paper is organized as follows: In Section 2 we recall some preliminaries for fully coupled FBSDEs with jumps, which will be used later. In Section 3, on one hand, we prove some basic estimates for fully coupled FBSDEs with jumps under monotonicity condition, on the other hand, assuming the Lipschitz coefficients of σ , h with respect to z , k to be sufficiently small, we establish the well-posedness result and a generalized Comparison Theorem for fully coupled FBSDEs with jumps on a small time interval. The associated L^p -estimates ($p \geq 2$) are then derived.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is a natural filtration generated by the following two mutually independent processes, and completed by all P -null sets:

- (i) a d -dimensional standard Brownian motion $\{B_t\}_{t \geq 0}$;
- (ii) a Poisson random measure μ on $\mathbb{R}^+ \times E$, where $E = \mathbb{R}^l \setminus \{0\}$ is equipped with its Borel σ -field $\mathcal{B}(E)$, with the compensator $\hat{\mu}(dt, de) = dt\lambda(de)$ such that $\{\hat{\mu}((0, t] \times A) = (\mu - \hat{\mu})((0, t] \times A)\}_{t \geq 0}$ being a martingale for all $A \in \mathcal{B}(E)$ satisfying $\lambda(A) < \infty$. Here λ is assumed to be a σ -finite Lévy measure on $(E, \mathcal{B}(E))$ with the property that $\int_E (1 \wedge |e|^2) \lambda(de) < \infty$.

For any $n \geq 1$, $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$. Fix $T > 0$, and $[0, T]$ is called the time duration. Now we give some spaces of processes which will be used later:

- $\mathcal{M}^2(t, T; \mathbb{R}^d) := \left\{ \varphi \mid \varphi : \Omega \times [t, T] \rightarrow \mathbb{R}^d \text{ is an } \mathbb{F}\text{-predictable process} : \|\varphi\|^2 = E[\int_t^T |\varphi_s|^2 ds] < +\infty \right\};$

- $\mathcal{S}^2(t, T; \mathbb{R}) := \left\{ \psi \mid \psi : \Omega \times [t, T] \rightarrow \mathbb{R} \text{ is an } \mathbb{F}\text{-adapted càdlàg process : } E\left[\sup_{t \leq s \leq T} |\psi_s|^2\right] < +\infty \right\};$
- $\mathcal{K}_\lambda^2(t, T; \mathbb{R}^n) := \left\{ K \mid K : \Omega \times [t, T] \times E \rightarrow \mathbb{R}^n \text{ is } \mathcal{P} \otimes \mathcal{B}(E) \text{ - measurable : } \right.$
 $\left. \|K\|^2 = E\left[\int_t^T \int_E |K_s(e)|^2 \lambda(de) ds\right] < +\infty \right\},$

where $t \in [0, T]$. Here \mathcal{P} denotes the σ -field of \mathbb{F} -predictable subsets of $\Omega \times [0, T]$.

2.1 Fully coupled FBSDEs with jumps

Now we consider the following fully coupled FBSDE with jumps associated with $(b, \sigma, h, f, \zeta, \Phi)$ on the time interval $[t, T]$ ($t \in [0, T]$):

$$\begin{cases} dX_s &= b(s, X_s, Y_s, Z_s, K_s)ds + \sigma(s, X_s, Y_s, Z_s, K_s)dB_s + \int_E h(s, X_{s-}, Y_{s-}, Z_s, K_s(e), e)\tilde{\mu}(dsde), \\ dY_s &= -f(s, X_s, Y_s, Z_s, \int_E K_s(e)l(e)\lambda(de))ds + Z_s dB_s + \int_E K_s(e)\tilde{\mu}(dsde), \quad s \in [t, T], \\ X_t &= \zeta, \\ Y_T &= \Phi(X_T), \end{cases} \quad (2.1)$$

where the solution (X, Y, Z, K) takes its values in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m$, and the coefficients

$$\begin{aligned} b &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^m) \longrightarrow \mathbb{R}^n, \\ \sigma &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^m) \longrightarrow \mathbb{R}^{n \times d}, \\ h &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m \times E \longrightarrow \mathbb{R}^n, \\ f &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m, \end{aligned}$$

$l : E \longrightarrow \mathbb{R}$ and $\Phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfy

(H2.1) (i) b, σ, f are uniformly Lipschitz with respect to (x, y, z, k) , and there exists $\rho : E \rightarrow \mathbb{R}^+$ with $\int_E \rho^2(e)\lambda(de) < +\infty$ such that, for any $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^n$, $y, \bar{y} \in \mathbb{R}^m$, $z, \bar{z} \in \mathbb{R}^{m \times d}$, $k, \bar{k} \in \mathbb{R}^m$ and $e \in E$,

$$|h(t, x, y, z, k, e) - h(t, \bar{x}, \bar{y}, \bar{z}, \bar{k}, e)| \leq \rho(e)(|x - \bar{x}| + |y - \bar{y}| + |z - \bar{z}|) + C|k - \bar{k}|;$$

(ii) $k \rightarrow f(t, x, y, z, k)$ is non-decreasing, for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$;

(iii) there exists a constant $C > 0$ such that

$$0 \leq l(e) \leq C(1 \wedge |e|), \quad x \in \mathbb{R}^n, \quad e \in E;$$

(iv) $\Phi(x)$ is uniformly Lipschitz with respect to $x \in \mathbb{R}^n$;

(v) for every $(x, y, z, k) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m$, $\Phi(x) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m)$, b, σ, h, f are \mathbb{F} -progressively measurable and

$$E \int_0^T |b(s, 0, 0, 0, 0)|^2 ds + E \int_0^T |f(s, 0, 0, 0, 0)|^2 ds + E \int_0^T |\sigma(s, 0, 0, 0, 0)|^2 ds + E \int_0^T \int_E |h(s, 0, 0, 0, 0, e)|^2 \lambda(de) ds < \infty.$$

Let

$$g(s, x, y, z, k) := f(s, x, y, z, \int_E k(e)l(e)\lambda(de)),$$

$$(s, x, y, z, k) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}).$$

In this paper we use the usual inner product and the Euclidean norm in \mathbb{R}^n , \mathbb{R}^m and $\mathbb{R}^{m \times d}$, respectively. Given an $m \times n$ full-rank matrix G , we define:

$$\pi = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A(t, \pi, k) = \begin{pmatrix} -G^T g \\ Gb \\ G\sigma \end{pmatrix}(t, \pi, k),$$

where G^T is the transposed matrix of G .

We assume the following monotonicity conditions:

- (H2.2) (i) $\langle A(t, \pi, k) - A(t, \bar{\pi}, \bar{k}), \pi - \bar{\pi} \rangle + \int_E \langle G\hat{h}(e), \hat{k}(e) \rangle \lambda(de)$
 $\leq -\beta_1 |G\hat{x}|^2 - \beta_2 (|G^T \hat{y}|^2 + |G^T \hat{z}|^2) - \beta_3 \int_E |G^T \hat{k}(e)|^2 \lambda(de),$
(ii) $\langle \Phi(x) - \Phi(\bar{x}), G(x - \bar{x}) \rangle \geq \mu_1 |G\hat{x}|^2, \forall \pi = (x, y, z), \bar{\pi} = (\bar{x}, \bar{y}, \bar{z}), \hat{x} = x - \bar{x}, \hat{y} = y - \bar{y}, \hat{z} = z - \bar{z}, \hat{k} = k - \bar{k}, \hat{h}(e) = h(t, \pi, k, e) - h(t, \bar{\pi}, \bar{k}, e),$
where $\beta_1, \beta_2, \beta_3, \mu_1$ are nonnegative constants with $\beta_1 + \beta_2 > 0, \beta_1 + \beta_3 > 0, \beta_2 + \mu_1 > 0, \beta_3 + \mu_1 > 0$.
Moreover, we have $\beta_1 > 0, \mu_1 > 0$ (resp., $\beta_2 > 0, \beta_3 > 0$), when $m > n$ (resp., $m < n$).

Remark 2.1. (H2.2)-(ii)' (H2.2) (ii) results in the weaker condition: $\langle \Phi(x) - \Phi(\bar{x}), G(x - \bar{x}) \rangle \geq 0$, for all $x, \bar{x} \in \mathbb{R}^n$.

When $\Phi(x) = \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m)$, (H2.2)-(i) can be weakened as follows:

- (H2.3) $\langle A(t, \pi, k) - A(t, \bar{\pi}, \bar{k}), \pi - \bar{\pi} \rangle + \int_E \langle G\hat{h}(e), \hat{k}(e) \rangle \lambda(de) \leq -\beta_1 |G\hat{x}|^2 - \beta_2 |G^T \hat{y}|^2,$
where β_1, β_2 are nonnegative constants with $\beta_1 + \beta_2 > 0$. Moreover, we have $\beta_1 > 0$ (resp., $\beta_2 > 0$), when $m > n$ (resp., $m < n$).

Lemma 2.1. Under the assumptions (H2.1) and (H2.2), for any $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, FBSDE (2.1) has a unique adapted solution $(X_s, Y_s, Z_s, K_s)_{s \in [t, T]} \in \mathcal{S}^2(t, T; \mathbb{R}^n) \times \mathcal{S}^2(t, T; \mathbb{R}^m) \times \mathcal{M}^2(t, T; \mathbb{R}^{m \times d}) \times \mathcal{K}_\lambda^2(t, T; \mathbb{R}^m)$.

Lemma 2.2. Under the assumptions (H2.2)-(ii)' and (H2.3), for any $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ and the terminal condition $\Phi(x) = \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m)$, FBSDE (2.1) has a unique adapted solution $(X_s, Y_s, Z_s, K_s)_{s \in [t, T]} \in \mathcal{S}^2(t, T; \mathbb{R}^n) \times \mathcal{S}^2(t, T; \mathbb{R}^m) \times \mathcal{M}^2(t, T; \mathbb{R}^{m \times d}) \times \mathcal{K}_\lambda^2(t, T; \mathbb{R}^m)$.

For the proof, the reader can refer to Wu [17, 18].

3 Regularity results for solutions of fully coupled FBSDEs with jumps

In this section we will study some important estimates for solutions of fully coupled FBSDEs with jumps.

3.1 Regularity results under the monotonicity condition

First, we derive some useful estimates for the solutions under the monotonicity condition.

Let now be given the mappings

$$\begin{aligned} b &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}) \longrightarrow \mathbb{R}^n, \\ \sigma &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}) \longrightarrow \mathbb{R}^d, \\ h &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^n, \\ g &: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}, \end{aligned}$$

and $\Phi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying (H2.1), (H2.2), and also assume

- (H3.1) For any $t \in [0, T]$, for any $(x, y, z, k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$, P-a.s.,

$$|b(t, x, y, z, k)| + |\sigma(t, x, y, z, k)| + |g(t, x, y, z, k)| + |\Phi(x)| \leq L(1 + |x| + |y| + |z| + |k|),$$

and there exists a measurable function $\rho : E \rightarrow \mathbb{R}^+$ with $\int_E \rho^2(e) \lambda(de) < +\infty$ such that, for any $t \in [0, T]$, $(x, y, z, k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ and $e \in E$,

$$|h(t, x, y, z, k, e)| \leq \rho(e)(1 + |x| + |y| + |z| + |k|).$$

We consider the following fully coupled FBSDE with jumps, parameterized by the initial condition $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$:

$$\begin{cases} dX_s^{t, \zeta} &= b(s, \Pi_s^{t, \zeta}, K_s^{t, \zeta}) ds + \sigma(s, \Pi_s^{t, \zeta}, K_s^{t, \zeta}) dB_s + \int_E h(s, \Pi_{s-}^{t, \zeta}, K_{s-}^{t, \zeta}(e), e) \tilde{\mu}(ds de), \\ dY_s^{t, \zeta} &= -g(s, \Pi_s^{t, \zeta}, K_s^{t, \zeta}) ds + Z_s^{t, \zeta} dB_s + \int_E K_s^{t, \zeta}(e) \tilde{\mu}(ds de), \quad s \in [t, T], \\ X_t^{t, \zeta} &= \zeta, \\ Y_T^{t, \zeta} &= \Phi(X_T^{t, \zeta}), \end{cases} \quad (3.1)$$

where we have put $\Pi_s^{t, \zeta} = (X_s^{t, \zeta}, Y_s^{t, \zeta}, Z_s^{t, \zeta})$, and $\Pi_{s-}^{t, \zeta} = (X_{s-}^{t, \zeta}, Y_{s-}^{t, \zeta}, Z_{s-}^{t, \zeta})$.

Proposition 3.1. *Under the assumptions (H2.1), (H2.2), (H3.1), for any $0 \leq t \leq T$ and any associated initial states $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have the following estimates, P-a.s.:*

- (i) $E[\sup_{t \leq s \leq T} |X_s^{t, \zeta} - X_s^{t, \zeta'}|^2 + \sup_{t \leq s \leq T} |Y_s^{t, \zeta} - Y_s^{t, \zeta'}|^2 + \int_t^T |Z_s^{t, \zeta} - Z_s^{t, \zeta'}|^2 ds + \int_t^T \int_E |K_s^{t, \zeta}(e) - K_s^{t, \zeta'}(e)|^2 \lambda(de) ds \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2,$
- (ii) $E[\sup_{t \leq s \leq T} |X_s^{t, \zeta}|^2 + \sup_{t \leq s \leq T} |Y_s^{t, \zeta}|^2 + \int_t^T |Z_s^{t, \zeta}|^2 ds + \int_t^T \int_E |K_s^{t, \zeta}(e)|^2 \lambda(de) ds \mid \mathcal{F}_t] \leq C(1 + |\zeta|^2).$

If σ, h also satisfy:

(H3.2) for any $t \in [0, T]$, for any $(x, y, z, k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, P-a.s., $|\sigma(t, x, y, z, k)| \leq L(1 + |x| + |y|)$, $|h(t, x, y, z, k, e)| \leq \rho(e)(1 + |x| + |y|)$, then we can get

- (iii) $E[\sup_{t \leq s \leq t+\delta} |X_s^{t, \zeta} - \zeta|^2 \mid \mathcal{F}_t] \leq C\delta(1 + |\zeta|^2)$, P-a.s., $0 \leq \delta \leq T - t$.

Proof. From Lemma 2.1, we know there exist the unique solutions $(\Pi^{t, \zeta}, K^{t, \zeta}) \in \mathcal{S}^2(t, T; \mathbb{R}^n) \times \mathcal{S}^2(t, T; \mathbb{R}) \times \mathcal{M}^2(t, T; \mathbb{R}^d) \times \mathcal{K}_\lambda^2(t, T; \mathbb{R})$, and $(\Pi^{t, \zeta'}, K^{t, \zeta'}) \in \mathcal{S}^2(t, T; \mathbb{R}^n) \times \mathcal{S}^2(t, T; \mathbb{R}) \times \mathcal{M}^2(t, T; \mathbb{R}^d) \times \mathcal{K}_\lambda^2(t, T; \mathbb{R})$ for FBSDE (3.1) associated with ζ and ζ' , respectively. For convenience, we define

$$\begin{aligned} \hat{X}_s &:= X_s^{t, \zeta} - X_s^{t, \zeta'}, \quad \hat{Y}_s := Y_s^{t, \zeta} - Y_s^{t, \zeta'}, \quad \hat{Z}_s := Z_s^{t, \zeta} - Z_s^{t, \zeta'}, \quad \hat{K}_s := K_s^{t, \zeta} - K_s^{t, \zeta'}, \\ \Delta l(s) &:= l(s, \Pi_s^{t, \zeta}, K_s^{t, \zeta}) - l(s, \Pi_s^{t, \zeta'}, K_s^{t, \zeta'}), \quad \Delta h(s, e) := h(s, \Pi_s^{t, \zeta}, K_s^{t, \zeta}(e), e) - h(s, \Pi_s^{t, \zeta'}, K_s^{t, \zeta'}(e), e), \end{aligned}$$

where $l = b, \sigma, g, A$, respectively.

Applying Itô's formula to $|\hat{X}_s|^2$, we obtain from the Gronwall inequality,

$$E[|\hat{X}_s|^2 \mid \mathcal{F}_t] \leq C(|\zeta - \zeta'|^2 + E[\int_t^s (|\hat{Y}_r|^2 + |\hat{Z}_r|^2 + \int_E |\hat{K}_r(e)|^2 \lambda(de)) dr \mid \mathcal{F}_t]), \quad t \leq s \leq T. \quad (3.2)$$

Then, applying Itô's formula to $e^{\beta s} |\hat{Y}_s|^2$, taking β large enough, and taking into account (3.2), we get

$$\begin{aligned} E[|\hat{Y}_s|^2 \mid \mathcal{F}_t] &+ E[\int_s^T |\hat{Y}_r|^2 dr + \int_s^T |\hat{Z}_r|^2 dr + \int_s^T \int_E |\hat{K}_r(e)|^2 \lambda(de) dr \mid \mathcal{F}_t] \\ &\leq C|\zeta - \zeta'|^2 + CE[\int_t^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2 + \int_E |\hat{K}_r(e)|^2 \lambda(de)) dr \mid \mathcal{F}_t], \quad t \leq s \leq T. \end{aligned} \quad (3.3)$$

On the other hand, applying Itô's formula to $\langle G\hat{X}_r, \hat{Y}_r \rangle$, from the assumption (H2.2) we get

$$\begin{aligned} \langle G\hat{X}_s, \hat{Y}_s \rangle &= E[\langle G\hat{X}_T, \hat{Y}_T \rangle \mid \mathcal{F}_s] - E[\int_s^T (\langle \Delta A(r), (\hat{X}_r, \hat{Y}_r, \hat{Z}_r) \rangle + \int_E \langle G\Delta h(r, e), \hat{K}_r(e) \rangle \lambda(de)) dr \mid \mathcal{F}_s] \\ &\geq E[\mu_1 |G\hat{X}_T|^2 \mid \mathcal{F}_s] + E[\beta_1 \int_s^T |G\hat{X}_r|^2 dr \mid \mathcal{F}_s] \\ &\quad + E[\int_s^T \beta_2 (|G^T \hat{Y}_r|^2 + |G^T \hat{Z}_r|^2) \mid \mathcal{F}_s] + E[\int_s^T \int_E \beta_3 |G^T \hat{K}_r(e)|^2 \lambda(de) dr \mid \mathcal{F}_s], \end{aligned} \quad (3.4)$$

Therefore, $\langle G\hat{X}_s, \hat{Y}_s \rangle \geq 0$, $t \leq s \leq T$, P-a.s.

If $\beta_2 > 0$, $\beta_3 > 0$, then we get

$$\begin{aligned} \langle G\hat{X}_t, \hat{Y}_t \rangle &= E[\langle G\hat{X}_s, \hat{Y}_s \rangle \mid \mathcal{F}_t] - E[\int_t^s (\langle \Delta A(r), (\hat{X}_r, \hat{Y}_r, \hat{Z}_r) \rangle + \int_E \langle G\Delta h(r, e), \hat{K}_r(e) \rangle \lambda(de)) dr \mid \mathcal{F}_t] \\ &\geq \beta_2 E[\int_t^s (|G^T \hat{Y}_r|^2 + |G^T \hat{Z}_r|^2) \mid \mathcal{F}_t] + \beta_3 E[\int_t^s \int_E |G^T \hat{K}_r(e)|^2 \lambda(de) dr \mid \mathcal{F}_t], \quad t \leq s \leq T, \text{ P-a.s.} \end{aligned} \quad (3.5)$$

Therefore, noticing here $m = 1$,

$$E[\int_t^s (|\hat{Y}_r|^2 + |\hat{Z}_r|^2 + \int_E |\hat{K}_r(e)|^2 \lambda(de)) dr \mid \mathcal{F}_t] \leq C \langle G\hat{X}_t, \hat{Y}_t \rangle, \quad t \leq s \leq T, \text{ P-a.s.} \quad (3.6)$$

Then, from (3.2) we can get

$$E[|\hat{X}_s|^2 \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2 + C \langle G\hat{X}_t, \hat{Y}_t \rangle, \quad t \leq s \leq T, \text{ P-a.s.} \quad (3.7)$$

From (3.3) we have

$$E[|\hat{Y}_s|^2 | \mathcal{F}_t] + E\left[\int_s^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2 + \int_E |\hat{K}_r(e)|^2 \lambda(de)) dr | \mathcal{F}_t\right] \leq C|\zeta - \zeta'|^2 + C\langle G\hat{X}_t, \hat{Y}_t \rangle, \quad t \leq s \leq T, \text{ P-a.s.} \quad (3.8)$$

Therefore,

$$|\hat{Y}_t|^2 \leq C|\zeta - \zeta'|^2 + C|\hat{X}_t||\hat{Y}_t| \leq C|\zeta - \zeta'|^2 + C|\hat{X}_t|^2 + \frac{1}{2}|\hat{Y}_t|^2, \text{ P-a.s.},$$

which means $|\hat{Y}_t| \leq C|\zeta - \zeta'|$, P-a.s. Then, from (3.7), (3.8), we can get

$$\begin{aligned} & E[|\hat{X}_s|^2 | \mathcal{F}_t] + E[|\hat{Y}_s|^2 | \mathcal{F}_t] + E\left[\int_s^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2 + \int_E |\hat{K}_r(e)|^2 \lambda(de)) dr | \mathcal{F}_t\right] \\ & \leq C|\zeta - \zeta'|^2, \quad t \leq s \leq T, \text{ P-a.s.} \end{aligned}$$

If $\beta_2 = 0$, $\beta_3 = 0$, then from assumption **(H2.2)**, we have $\beta_1 > 0$, $\mu_1 > 0$, $m = n = 1$, i.e. $G \in \mathbb{R} \setminus \{0\}$. From (3.4),

$$E[|\hat{X}_T|^2 | \mathcal{F}_t] + E\left[\int_t^T |\hat{X}_r|^2 dr | \mathcal{F}_t\right] \leq CG\hat{X}_t \cdot \hat{Y}_t, \quad C > 0.$$

From (3.3) combined with (3.5),

$$|\hat{Y}_t|^2 + E\left[\int_t^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2 + \int_E |\hat{K}_r(e)|^2 \lambda(de)) dr | \mathcal{F}_t\right] \leq CG\hat{X}_t \cdot \hat{Y}_t \leq C|\zeta - \zeta'|^2 + \frac{1}{2}|\hat{Y}_t|^2.$$

Therefore,

$$|\hat{Y}_t|^2 + E\left[\int_t^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2 + \int_E |\hat{K}_r(e)|^2 \lambda(de)) dr | \mathcal{F}_t\right] \leq C|\zeta - \zeta'|^2.$$

Furthermore, from (3.2),

$$E[|\hat{X}_s|^2 | \mathcal{F}_t] \leq C|\zeta - \zeta'|^2, \quad t \leq s \leq T, \text{ P-a.s.}$$

Therefore,

$$\begin{aligned} E\left[\sup_{t \leq s \leq T} |\hat{X}_s|^2 | \mathcal{F}_t\right] & \leq 3|\zeta - \zeta'|^2 + CE\left[\int_t^T |\Delta b(r)|^2 dr + \int_t^T |\Delta \sigma(r)|^2 dr + \int_t^T \int_E |\Delta h(r, e)|^2 \lambda(de) dr | \mathcal{F}_t\right] \\ & \leq 3|\zeta - \zeta'|^2 + CE\left[\int_t^T (|\hat{X}_r|^2 + |\hat{Y}_r|^2 + |\hat{Z}_r|^2 + \int_E |\hat{K}_r(e)|^2 \lambda(de)) dr | \mathcal{F}_t\right] \\ & \leq C|\zeta - \zeta'|^2, \text{ P-a.s.}; \end{aligned}$$

similarly, we have

$$\begin{aligned} E\left[\sup_{t \leq s \leq T} |\hat{Y}_s|^2 | \mathcal{F}_t\right] & \leq CE[|\hat{X}_T|^2 | \mathcal{F}_t] + CE\left[\int_t^T (|\hat{X}_r|^2 + |\hat{Y}_r|^2 + |\hat{Z}_r|^2 + \int_E |\hat{K}_r(e)|^2 \lambda(de)) dr | \mathcal{F}_t\right] \\ & \leq C|\zeta - \zeta'|^2, \text{ P-a.s.} \end{aligned}$$

In this way, we complete the proof of (i). Also, (ii) can be proved similarly by making full use of the monotonic assumption **(H2.2)**. For (iii), similarly, using **(H3.2)**,

$$\begin{aligned} & E\left[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta} - \zeta|^2 | \mathcal{F}_t\right] \\ & \leq 2E\left[\int_t^{t+\delta} |b(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta})|^2 dr | \mathcal{F}_t\right] + CE\left[\int_t^{t+\delta} |\sigma(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta})|^2 dr | \mathcal{F}_t\right] \\ & \quad + CE\left[\int_t^{t+\delta} \int_E |h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_{r-}^{t,\zeta}, K_{r-}^{t,\zeta}, e)|^2 \lambda(de) dr | \mathcal{F}_t\right] \\ & \leq C\delta E\left[\int_t^{t+\delta} (1 + |X_r^{t,\zeta}|^2 + |Y_r^{t,\zeta}|^2 + |Z_r^{t,\zeta}|^2 + \int_E |K_r^{t,\zeta}(e)|^2 \lambda(de)) dr | \mathcal{F}_t\right] \\ & \quad + CE\left[\int_t^{t+\delta} (1 + |X_r^{t,\zeta}|^2 + |Y_r^{t,\zeta}|^2) dr | \mathcal{F}_t\right] \\ & \leq C\delta E\left[\sup_{t \leq r \leq t+\delta} (|X_r^{t,\zeta}|^2 + |Y_r^{t,\zeta}|^2) + \int_t^{t+\delta} (|Z_r^{t,\zeta}|^2 + \int_E |K_r^{t,\zeta}(e)|^2 \lambda(de)) dr | \mathcal{F}_t\right] + C\delta \\ & \quad + C\delta E\left[\sup_{t \leq r \leq t+\delta} (|X_r^{t,\zeta}|^2 + |Y_r^{t,\zeta}|^2) | \mathcal{F}_t\right] \\ & \leq C\delta(1 + |\zeta|^2). \end{aligned}$$

□

Remark 3.1. From Proposition 3.1, we have, immediately,

$$|Y_t^{t,\zeta}| \leq C(1 + |\zeta|); \quad |Y_t^{t,\zeta} - Y_t^{t,\zeta'}| \leq C|\zeta - \zeta'|, \quad P\text{-a.s.}, \quad (3.9)$$

where the constant $C > 0$ depends only on the Lipschitz constants of b , σ , h , g and Φ .

Now we introduce the random field:

$$u(t, x) = Y_s^{t,x} \big|_{s=t}, \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

where $Y^{t,x}$ is the solution of FBSDE (3.1) with the initial state $x \in \mathbb{R}^n$.

From Remark 3.1, it is easy to check that, for all $t \in [0, T]$, P -a.s.,

$$\begin{aligned} \text{(i)} \quad & |u(t, x) - u(t, y)| \leq C|x - y|, \text{ for all } x, y \in \mathbb{R}^n; \\ \text{(ii)} \quad & |u(t, x)| \leq C(1 + |x|), \text{ for all } x \in \mathbb{R}^n. \end{aligned} \quad (3.10)$$

Remark 3.2. Moreover, it is well known that, under the additional assumption that the functions

$$b, \sigma, h, g \text{ and } \Phi \text{ are deterministic,}$$

also u is a deterministic function of (t, x) .

The random field u and $Y^{t,\zeta}$, $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, are related by the following theorem.

Theorem 3.1. Under the assumptions (H2.1), (H2.2), for any $t \in [0, T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have

$$u(t, \zeta) = Y_t^{t,\zeta}, \quad P\text{-a.s.}$$

The proof of Theorem 3.1 is similar to Theorem A.1 in [5] for the decoupled FBSDE with jumps, or Theorem 6.1 in [1].

Remark 3.3. (i) From Theorem 3.1, obviously, $Y_s^{t,\zeta} = Y_s^{s,X_s^{t,\zeta}} = u(s, X_s^{t,\zeta})$.

(ii) From now for convenience, we take $\rho(e) = C(1 \wedge |e|)$, where C is a constant.

Proposition 3.2. Under the assumptions (H2.1), (H2.2), (H3.1), (H3.2), for any $p \geq 2$, $0 \leq t \leq T$ and the associated initial states $\zeta, \zeta' \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, there exists $\tilde{\delta}_0 > 0$ which depends on p and the Lipschitz constant and the linear growth constant L , such that

$$\begin{aligned} \text{(i)} \quad & E \left[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta}|^p + \sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p + \left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds \right)^{\frac{p}{2}} \right. \\ & \left. + \left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds \right)^{\frac{p}{2}} \mid \mathcal{F}_t \right] \leq C_p(1 + |\zeta|^p), \quad P\text{-a.s.}; \\ \text{(ii)} \quad & E \left[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t \right] \leq C_p \delta (1 + |\zeta|^p), \quad P\text{-a.s.}, \quad 0 \leq \delta \leq \tilde{\delta}_0. \end{aligned}$$

Remark 3.4. Let us point out that, unlike FBSDEs without jumps, estimates (ii) does not hold true with $\delta^{\frac{p}{2}}$ instead of δ at the right hand, that is, one can't get the following estimate like FBSDEs without jumps, even for the decoupled FBSDEs with jumps: for all $p \geq 2$, $E \left[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t \right] \leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p)$, P -a.s., $0 \leq \delta \leq \delta_0$.

Indeed, if the above estimate is true, then one can get, for all $t \leq s \leq s + \delta \leq t + \delta_0$,

$$E[|X_{s+\delta}^{t,\zeta} - X_s^{t,\zeta}|^p] \leq E[E[|X_{s+\delta}^{s,X_s^{t,\zeta}} - X_s^{t,\zeta}|^p \mid \mathcal{F}_s]] \leq C_p \delta^{\frac{p}{2}} E[(1 + |X_s^{t,\zeta}|^p)],$$

and for $\frac{p}{2} > 2$, Kolmogorov's Continuity Criterion would imply the continuity of the jump process $X^{t,\zeta}$ which is impossible.

In order to prove Proposition 3.2, we need the following lemma.

Lemma 3.1. Under the assumptions (H2.1), (H2.2), (H3.1), (H3.2). For any $p \geq 2$,

$$E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds)^{\frac{p}{2}} | \mathcal{F}_t] \leq (\frac{p}{2})^{\frac{p}{2}} E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \mu(dsde))^{\frac{p}{2}} | \mathcal{F}_t]. \quad (3.11)$$

Proof. Setting $f_s := \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de)$, we have

$$\begin{aligned} & E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds)^{\frac{p}{2}} | \mathcal{F}_t] = E[(\int_t^{t+\delta} f_s ds)^{\frac{p}{2}} | \mathcal{F}_t] = \frac{p}{2} E[(\int_t^{t+\delta} f_s (\int_t^s f_r dr)^{\frac{p}{2}-1} ds) | \mathcal{F}_t] \\ & = \frac{p}{2} E[(\int_t^{t+\delta} \int_E \{(\int_t^s f_r dr)^{\frac{p}{2}-1} \cdot |K_s^{t,\zeta}(e)|^2\} \lambda(de) ds | \mathcal{F}_t] \\ & = \frac{p}{2} E[(\int_t^{t+\delta} \int_E \{(\int_t^s f_r dr)^{\frac{p}{2}-1} \cdot |K_s^{t,\zeta}(e)|^2\} \mu(dsde) | \mathcal{F}_t] \\ & \leq \frac{p}{2} E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \mu(dsde)) (\int_t^{t+\delta} f_r dr)^{\frac{p}{2}-1} | \mathcal{F}_t] \\ & \leq \frac{p}{2} (E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \mu(dsde))^{\frac{p}{2}} | \mathcal{F}_t])^{\frac{2}{p}} (E[(\int_t^{t+\delta} f_r dr)^{\frac{p}{2}} | \mathcal{F}_t])^{1-\frac{2}{p}}. \end{aligned}$$

Therefore, we have (3.11) if $E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds)^{\frac{p}{2}} | \mathcal{F}_t] < +\infty$, P-a.s. Otherwise, we approximate $|K_s^{t,\zeta}(e)|^2$ from below by an increasing sequence K^n of non-negative predictable functions over $\Omega \times [0, T] \times E$ such that $E[(\int_t^{t+\delta} \int_E |K_s^n(e)|^2 \lambda(de) ds)^{\frac{p}{2}} | \mathcal{F}_t] < +\infty$, $n \geq 1$. Then, with the same arguments as above we have

$$E[(\int_t^{t+\delta} \int_E |K_s^n(e)|^2 \lambda(de) ds)^{\frac{p}{2}} | \mathcal{F}_t] \leq (\frac{p}{2})^{\frac{p}{2}} E[(\int_t^{t+\delta} \int_E |K_s^n(e)|^2 \mu(dsde))^{\frac{p}{2}} | \mathcal{F}_t], \quad n \geq 1,$$

and taking the limit as $n \rightarrow +\infty$ by using the monotone convergence theorem, we obtain (3.11). \square

Now we give the proof of Proposition 3.2.

Proof. Without loss of generality, we restrict ourselves to the proof for $p = 2k$, $k \in \mathbb{Z}^+$.

From the Remarks 3.1 and 3.3 we have $|Y_s^{t,\zeta}| = |Y_s^{s, X_s^{t,\zeta}}| \leq C(1 + |X_s^{t,\zeta}|)$, P-a.s. Since

$$Y_t^{t,\zeta} = Y_s^{t,\zeta} + \int_t^s g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta}) dr - \int_t^s Z_r^{t,\zeta} dB_r - \int_t^s \int_E K_r^{t,\zeta}(e) \tilde{\mu}(drde), \quad t \leq s \leq t + \delta,$$

we get from Burkholder-Davis-Gundy inequality and (3.11),

$$\begin{aligned} & E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} | \mathcal{F}_t] + E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds)^{\frac{p}{2}} | \mathcal{F}_t] \\ & \leq E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} | \mathcal{F}_t] + C_p E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \mu(dsde))^{\frac{p}{2}} | \mathcal{F}_t] \\ & \leq C_p E[\sup_{t \leq s \leq t+\delta} |\int_t^s Z_r^{t,\zeta} dB_r + \int_t^s \int_E K_r^{t,\zeta}(e) \tilde{\mu}(drde)|^p | \mathcal{F}_t] \\ & \leq C_p E[\sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p + (\int_t^{t+\delta} |g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}, K_s^{t,\zeta})| ds)^p | \mathcal{F}_t] \\ & \leq C_p E[\sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p | \mathcal{F}_t] + C_p E[(\int_t^{t+\delta} (1 + |X_s^{t,\zeta}| + |Y_s^{t,\zeta}| + |Z_s^{t,\zeta}| + \int_E |K_s^{t,\zeta}(e)| (1 \wedge |e|) \lambda(de)) ds)^p | \mathcal{F}_t] \\ & \leq C_p E[\sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p | \mathcal{F}_t] + C_p \delta^p + C_p E[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta}|^p + \sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p | \mathcal{F}_t] \delta^p \\ & \quad + C_p E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} | \mathcal{F}_t] \delta^{\frac{p}{2}} + C_p E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds)^{\frac{p}{2}} | \mathcal{F}_t] \delta^{\frac{p}{2}} \\ & = C_p \delta^p + C_p \delta^p E[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta}|^p | \mathcal{F}_t] + (C_p + C_p \delta^p) E[\sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p | \mathcal{F}_t] \\ & \quad + C_p \delta^{\frac{p}{2}} E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} | \mathcal{F}_t] + C_p \delta^{\frac{p}{2}} E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds)^{\frac{p}{2}} | \mathcal{F}_t]. \end{aligned}$$

Choosing $\delta_0 > 0$, such that $1 - C_p \delta_0^{\frac{p}{2}} > 0$, we get, for any $0 \leq \delta \leq \delta_0$,

$$\begin{aligned} & E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} | \mathcal{F}_t] + E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds)^{\frac{p}{2}} | \mathcal{F}_t] \\ & \leq C_p \delta^p + C_p \delta^p E[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta}|^p | \mathcal{F}_t] + (C_p + C_p \delta^p) E[\sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p | \mathcal{F}_t]. \end{aligned} \quad (3.12)$$

On the other hand, from Remark 3.3 and (3.10), for $t \leq s \leq T$,

$$\begin{aligned}
& E[\sup_{t \leq r \leq s} |X_r^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t] \\
& \leq C_p E[(\int_t^s |b(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta})| dr)^p \mid \mathcal{F}_t] + C_p E[(\int_t^s |\sigma(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta})|^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t] \\
& \quad + C_p E[(\int_t^s \int_E |h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta}(e), e)|^2 \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t] \\
& \leq C_p E[(\int_t^s (1 + |X_r^{t,\zeta} - \zeta| + |\zeta| + |Z_r^{t,\zeta}| + \int_E |K_r^{t,\zeta}(e)| \lambda(de)) dr)^p \mid \mathcal{F}_t] \\
& \quad + C_p E[(\int_t^s (1 + |X_r^{t,\zeta}| + |Y_r^{t,\zeta}|)^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t] \\
& \quad + C_p E[(\int_t^s \int_E |h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta}(e), e)|^2 \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t] \\
& \leq C_p (1 + |\zeta|^p) (s - t)^{\frac{p}{2}} + C_p (s - t)^{\frac{p}{2}} E[(\int_t^s |Z_r^{t,\zeta}|^2 dr)^{\frac{p}{2}} + (\int_t^s \int_E |K_r^{t,\zeta}(e)|^2 \lambda(de) dr)^{\frac{p}{2}} \mid \mathcal{F}_t] \\
& \quad + C_p E[\int_t^s |X_r^{t,\zeta} - \zeta|^p dr \mid \mathcal{F}_t] + C_p E[(\int_t^s \int_E |h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta}(e), e)|^2 \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t],
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
& E[(\int_t^s \int_E |h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta}(e), e)|^2 \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t] \\
& \leq E[(\int_t^{t+\delta} \int_E C(1 \wedge |e|^2)(1 + |X_{r-}^{t,\zeta}| + |Y_{r-}^{t,\zeta}|)^2 \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t] \\
& \leq C_p E[(\int_t^{t+\delta} \int_E (1 \wedge |e|^2)(1 + |X_{r-}^{t,\zeta}|^2 + |Y_{r-}^{t,\zeta}|^2) \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t] \\
& \leq C_p E[(\int_t^{t+\delta} \int_E (1 \wedge |e|^2)(1 + |X_{r-}^{t,\zeta} - \zeta|^2 + |\zeta|^2) \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t] \\
& \leq C_p E[(\int_t^{t+\delta} \int_E (1 \wedge |e|^2) \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t] (1 + |\zeta|^p) + C_p E[(\int_t^{t+\delta} \int_E (1 \wedge |e|^2) |X_{r-}^{t,\zeta} - \zeta|^2 \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t].
\end{aligned} \tag{3.14}$$

Notice that

$$E[(\int_t^{t+\delta} \int_E (1 \wedge |e|^2) |X_{r-}^{t,\zeta} - \zeta|^2 \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t] \leq C_p \delta E[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t]. \tag{3.15}$$

Indeed, we denote $\tilde{X}_{s-}^{t,\zeta} := X_{s-}^{t,\zeta} - \zeta$, $\rho_s(e) := (1 \wedge |e|^2) |\tilde{X}_s^{t,\zeta}|^2$, $\bar{\rho}_s(e) := |\tilde{X}_s^{t,\zeta}|^2$, $A_r := \int_t^r \int_E \rho_s(e) \mu(dsde)$. Then, from Young inequality we have

$$\begin{aligned}
A_r^p - A_{t-}^p &= \sum_{t \leq s \leq r} (A_s^p - A_{s-}^p) = \sum_{t \leq s \leq r} ((\int_t^s \int_E \rho_r(e') \mu(drde'))^p - (\int_t^{s-} \int_E \rho_r(e') \mu(drde'))^p) \\
&= \sum_{t \leq s \leq r} \int_E ((\int_t^{s-} \int_E \rho_r(e') \mu(drde') + \rho_s(e))^p - (\int_t^{s-} \int_E \rho_r(e') \mu(dsde'))^p) \mu(\{s\}, de) \\
&= \sum_{t \leq s \leq r} \int_E \sum_{l=1}^p \binom{p}{l} (\int_t^{s-} \int_E \rho_r(e') \mu(drde'))^{p-l} \rho_s(e)^l \mu(\{s\}, de) \\
&= \int_t^r \int_E \sum_{l=1}^p \binom{p}{l} (\int_t^{s-} \int_E \rho_r(e') \mu(drde'))^{p-l} \rho_s(e)^l \mu(dsde) \\
&\leq C_p \int_t^r \int_E (1 \wedge |e|^2) ((\int_t^{s-} \int_E \rho_r(e') \mu(drde'))^p + |\bar{\rho}_s(e)|^p) \mu(dsde).
\end{aligned}$$

Therefore,

$$E[|\int_t^r \int_E \rho_s(e) \mu(dsde)|^p \mid \mathcal{F}_t] \leq C_p E[\int_t^r \int_E (1 \wedge |e|^2) ((\int_t^{s-} \int_E \rho_r(e') \mu(drde'))^p + |\bar{\rho}_s(e)|^p) \lambda(de) ds \mid \mathcal{F}_t].$$

From the Gronwall inequality, we get

$$E[|\int_t^r \int_E \rho_s(e) \mu(dsde)|^p \mid \mathcal{F}_t] \leq C_p E[\int_t^r \int_E (1 \wedge |e|^2) |\bar{\rho}_s(e)|^p \lambda(de) ds \mid \mathcal{F}_t].$$

Therefore,

$$E[|\int_t^r \int_E |X_{s-}^{t,\zeta} - \zeta|^2 (1 \wedge |e|^2) \mu(dsde)|^p \mid \mathcal{F}_t] \leq C_p E[\int_t^r |X_{s-}^{t,\zeta} - \zeta|^{2p} ds \mid \mathcal{F}_t] \leq C_p (r - t) E[\sup_{t \leq s \leq r} |X_{s-}^{t,\zeta} - \zeta|^{2p} \mid \mathcal{F}_t].$$

Similarly, $E[|\int_t^r \int_E (1 \wedge |e|^2) \mu(dsde)|^{\frac{p}{2}} \mid \mathcal{F}_t] \leq C_p (r - t)$. Thus, from (3.14) we have

$$E[(\int_t^s \int_E |h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta}(e), e)|^2 \mu(drde))^{\frac{p}{2}} \mid \mathcal{F}_t] \leq C_p \delta (1 + |\zeta|^p) + C_p \delta E[\sup_{t \leq r \leq s} |X_r^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t]. \tag{3.16}$$

Consequently, from (3.13),

$$\begin{aligned} E\left[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t\right] &\leq C_p \delta (1 + |\zeta|^p) + C_p \delta E\left[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t\right] \\ &\quad + C_p \delta^{\frac{p}{2}} E\left[\left(\int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr\right)^{\frac{p}{2}} + \left(\int_t^{t+\delta} \int_E |K_r^{t,\zeta}(e)|^2 \lambda(de) dr\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right], \text{ P-a.s.} \end{aligned} \quad (3.17)$$

Choosing $\delta_1 > 0$, such that $1 - C_p \delta_1 > 0$, for any $0 \leq \delta \leq \delta_1$, we have

$$\begin{aligned} &E\left[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t\right] \\ &\leq C_p \delta (1 + |\zeta|^p) + C_p \delta^{\frac{p}{2}} E\left[\left(\int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr\right)^{\frac{p}{2}} + \left(\int_t^{t+\delta} \int_E |K_r^{t,\zeta}(e)|^2 \lambda(de) dr\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right], \text{ P-a.s.} \end{aligned} \quad (3.18)$$

Then, from (3.12), (3.18) and $|Y_s^{t,\zeta}| \leq C(1 + |X_s^{t,\zeta}|)$, we have

$$\begin{aligned} &E\left[\left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] + E\left[\left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] \\ &\leq C_p \delta^p (1 + |\zeta|^p) + C_p \delta^p + C_p + (C_p + C_p \delta^p) E\left[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t\right] \\ &\leq C_p \delta^p + C_p + C_p \delta (1 + |\zeta|^p) \\ &\quad + (C_p + C_p \delta^p) C_p \delta^{\frac{p}{2}} (E\left[\left(\int_t^s |Z_r^{t,\zeta}|^2 dr\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] + E\left[\left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right]), \end{aligned}$$

and taking $0 < \tilde{\delta}_0 \leq \min(\delta_0, \delta_1)$ such that $1 - (C_p + C_p \tilde{\delta}_0^p) C_p \tilde{\delta}_0^{\frac{p}{2}} > 0$, we have for all $0 \leq \delta \leq \tilde{\delta}_0$,

$$E\left[\left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] + E\left[\left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] \leq C_p (1 + |\zeta|^p), \text{ P-a.s.}$$

From (3.18), we get

$$E\left[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t\right] \leq C_p \delta (1 + |\zeta|^p), \text{ P-a.s., } 0 \leq \delta \leq \tilde{\delta}_0.$$

Hence, finally, from $|Y_s^{t,\zeta}| \leq C(1 + |X_s^{t,\zeta}|)$, we have

$$E\left[\sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p \mid \mathcal{F}_t\right] \leq C_p (1 + |\zeta|^p), \text{ P-a.s., } 0 \leq \delta \leq \tilde{\delta}_0.$$

□

3.2 Well-posedness and regularity results of fully coupled FBSDEs with jumps on the small time interval

In this subsection, we first prove that the fully coupled FBSDEs with jumps have a unique solution on a small time interval, if the Lipschitz coefficients of σ , h with respect to z , k are sufficiently small. Then, under these assumptions, we prove some regularity results for the solutions of fully coupled FBSDEs with jumps.

Theorem 3.2. *We suppose the assumptions (H2.1), (H3.1), (H3.3) hold true, where assumption (H3.3) is the following:*

(H3.3) *The Lipschitz constant $L_\sigma \geq 0$ of σ with respect to z , k is sufficiently small, i.e., there exists some $L_\sigma \geq 0$ small enough such that, for all $t \in [0, T]$, $x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$, $k_1, k_2 \in \mathbb{R}$,*

$$|\sigma(t, x_1, y_1, z_1, k_1) - \sigma(t, x_2, y_2, z_2, k_2)| \leq K(|x_1 - x_2| + |y_1 - y_2|) + L_\sigma(|z_1 - z_2| + |k_1 - k_2|).$$

Also the Lipschitz coefficient $L_h(\cdot)$ of h with respect to z , k is sufficiently small, i.e., there exists a function $L_h : E \rightarrow \mathbb{R}^+$ with $\tilde{C}_h := \max(\sup_{e \in E} L_h^2(e), \int_E L_h^2(e) \lambda(de)) < +\infty$ sufficiently small, and for all

$t \in [0, T]$, $x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$, $k_1, k_2 \in \mathbb{R}$, $e \in E$,

$$|h(t, x_1, y_1, z_1, k_1, e) - h(t, x_2, y_2, z_2, k_2, e)| \leq \rho(e)(|x_1 - x_2| + |y_1 - y_2|) + L_h(e)(|z_1 - z_2| + |k_1 - k_2|).$$

Then, there exists a constant $\delta_0 > 0$ only depending on the Lipschitz constants K and L_σ , \tilde{C}_h , such that, for every $0 \leq \delta \leq \delta_0$, and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, FBSDE (3.1) has a unique solution $(\Pi_s^{t,\zeta}, K_s^{t,\zeta})_{s \in [t, t+\delta]}$ on the time interval $[t, t+\delta]$.

Proof. It is easy to see, for any $v = ((y, z), k) \in \mathcal{M}^2(t, T; \mathbb{R}^{1+d}) \times K_\lambda^2(t, T; \mathbb{R})$, there exists a unique solution $V = ((Y, Z), K) \in \mathcal{M}^2(t, T; \mathbb{R}^{1+d}) \times K_\lambda^2(t, T; \mathbb{R})$ to the following decoupled FBSDE with jumps:

$$\begin{cases} dX_s &= b(s, X_s, y_s, z_s, k_s)ds + \sigma(s, X_s, y_s, z_s, k_s)dB_s + \int_E h(s, X_{s-}, y_{s-}, z_s, k_s(e), e)\tilde{\mu}(dsde), \\ dY_s &= -g(s, X_s, Y_s, Z_s, K_s)ds + Z_sdB_s + \int_E K_s(e)\tilde{\mu}(dsde), \quad s \in [t, T], \\ X_t &= \zeta, \\ Y_T &= \Phi(X_T). \end{cases} \quad (3.19)$$

We will prove that there exists a constant $\delta_0 > 0$, only depending on the Lipschitz constants K , L_σ and $L_h(\cdot)$ such that for every $0 \leq \delta \leq \delta_0$ the following mapping

$$I : \mathcal{M}^2(t, t+\delta; \mathbb{R}^{1+d}) \times K_\lambda^2(t, t+\delta; \mathbb{R}) \rightarrow \mathcal{M}^2(t, t+\delta; \mathbb{R}^{1+d}) \times K_\lambda^2(t, t+\delta; \mathbb{R})$$

is a contraction. Let $v_i = ((y_i, z_i), k_i) \in \mathcal{M}^2(t, t+\delta; \mathbb{R}^{1+d}) \times K_\lambda^2(t, t+\delta; \mathbb{R})$, and $V_i = I(v_i)$, $i = 1, 2$. We define $\hat{v} = ((y_1 - y_2, z_1 - z_2), k_1 - k_2)$, and $\hat{V} = ((Y_1 - Y_2, Z_1 - Z_2), K_1 - K_2)$, $\hat{X} = X_1 - X_2$. Then, by the usual techniques and the Gronwall inequality, we get

$$\begin{aligned} & E[\sup_{t \leq s \leq T} |\hat{X}_s|^2 | \mathcal{F}_t] \\ & \leq CE[\int_t^T |\hat{y}_s|^2 ds | \mathcal{F}_t] + C((T-t) + L_\sigma^2 + \int_E L_h^2(e)\lambda(de) + \sup_{e \in E} L_h^2(e))E[\int_t^T (|\hat{z}_s|^2 + \int_E |\hat{k}_s(e)|^2 \lambda(de))ds | \mathcal{F}_t] \\ & \leq C(T-t)E[\sup_{t \leq s \leq T} |\hat{y}_s|^2 | \mathcal{F}_t] + C((T-t) + L_\sigma^2 + \tilde{C}_h)E[\int_t^T (|\hat{z}_s|^2 + \int_E |\hat{k}_s(e)|^2 \lambda(de))ds | \mathcal{F}_t]. \end{aligned} \quad (3.20)$$

On the other hand, by using BSDE standard estimate, combined with (3.20), we get

$$\begin{aligned} & E[\sup_{t \leq s \leq T} |\hat{Y}_s|^2 + \int_t^T |\hat{Z}_s|^2 ds + \int_t^T \int_E |\hat{K}_s(e)|^2 \lambda(de)ds] \\ & \leq CE[|\Phi(X_T^1) - \Phi(X_T^2)|^2] + CE[\int_t^T |g(r, X_r^1, V_r^1) - g(r, X_r^2, V_r^2)|^2 dr] \\ & \leq CE[|\hat{X}_T|^2] + CE[\int_t^T |\hat{X}_r|^2 dr] \\ & \leq C(T-t)E[\sup_{t \leq s \leq T} |\hat{y}_s|^2] + C((T-t) + L_\sigma^2 + \tilde{C}_h)E[\int_t^T (|\hat{z}_s|^2 + \int_E |\hat{k}_s(e)|^2 \lambda(de))ds] \\ & \leq C((T-t) + L_\sigma^2 + \tilde{C}_h)(E[\sup_{t \leq s \leq T} |\hat{y}_s|^2] + E[\int_t^T (|\hat{z}_s|^2 + \int_E |\hat{k}_s(e)|^2 \lambda(de))ds]). \end{aligned}$$

As L_σ , \tilde{C}_h are sufficiently small, there exists $\delta_0 > 0$ such that $C\delta_0 + CL_\sigma^2 + C\tilde{C}_h < \frac{1}{2}$, and therefore, for any $0 \leq \delta \leq \delta_0$, we have

$$\begin{aligned} & E[\sup_{t \leq s \leq t+\delta} |\hat{Y}_s|^2 + \int_t^{t+\delta} |\hat{Z}_s|^2 ds + \int_t^T \int_E |\hat{K}_s(e)|^2 \lambda(de)ds] \\ & \leq \frac{1}{2}(E[\sup_{t \leq s \leq t+\delta} |\hat{y}_s|^2 + \int_t^{t+\delta} |\hat{z}_s|^2 ds + \int_t^T \int_E |\hat{k}_s(e)|^2 \lambda(de)ds]), \end{aligned} \quad (3.21)$$

which means, for any $0 \leq \delta \leq \delta_0$ this mapping I has a unique fixed point $I(V) = V$, i.e., FBSDE (3.1) has a unique solution $(\Lambda_s^{t,\zeta}, K_s^{t,\zeta})_{s \in [t, t+\delta]} := (X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}, K_s^{t,\zeta})_{s \in [t, t+\delta]}$ on $[t, t+\delta]$. \square

Remark 3.5. In fact, from the proof we see that L_σ , $\tilde{C}_h \geq 0$ such that $CL_\sigma^2 + C\tilde{C}_h < 1$ is sufficient for Proposition 3.2.

Next we will prove a comparison theorem for the following fully coupled FBSDE with jumps:

$$\begin{cases} dX_s &= b(s, X_s, Y_s, Z_s)ds + \sigma(s, X_s, Y_s, Z_s)dB_s + \int_E h(s, X_{s-}, Y_{s-}, Z_s, e)\tilde{\mu}(dsde), \\ dY_s &= -g(s, X_s, Y_s, Z_s, K_s)ds + Z_sdB_s + \int_E K_s(e)\tilde{\mu}(dsde), \quad s \in [t, t+\delta], \\ X_t &= \zeta, \\ Y_{t+\delta} &= \Phi(X_{t+\delta}). \end{cases} \quad (3.22)$$

Theorem 3.3. (Generalized Comparison Theorem) We suppose that the assumptions **(H2.1)**, **(H3.1)**, **(H3.3)** are satisfied. Let $\delta_0 > 0$ be a constant, only depending on the Lipschitz constants K , L_σ and $L_h(\cdot)$, such that for every $0 \leq \delta \leq \delta_0$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, FBSDE (3.22) has a unique solution $(X_s^i, Y_s^i, Z_s^i, K_s^i)_{s \in [t, t+\delta]}$ associated with $(b, \sigma, g, \zeta, \Phi_i)$ on the time interval $[t, t+\delta]$, respectively. Then, if for any $0 \leq \delta \leq \delta_0$ it holds $\Phi_1(X_{t+\delta}^2) \geq \Phi_2(X_{t+\delta}^2)$, P-a.s. (resp., $\Phi_1(X_{t+\delta}^1) \geq \Phi_2(X_{t+\delta}^1)$, P-a.s.), we also get $Y_t^1 \geq Y_t^2$, P-a.s.

The proof is similar to that of Theorem 4.1 in Wu [18]; we sketch it. For notational simplification, we assume $d = n = 1$.

Proof. We define $(\hat{X}, \hat{Y}, \hat{Z}, \hat{K}) := (X^1 - X^2, Y^1 - Y^2, Z^1 - Z^2, K^1 - K^2)$. Then $(\hat{X}, \hat{Y}, \hat{Z}, \hat{K})$ satisfies the following FBSDE:

$$\begin{cases} d\hat{X}_s &= (b_s^1 \hat{X}_s + b_s^2 \hat{Y}_s + b_s^3 \hat{Z}_s)ds + (\sigma_s^1 \hat{X}_s + \sigma_s^2 \hat{Y}_s + \sigma_s^3 \hat{Z}_s)dB_s + \int_E (h_s^1 \hat{X}_{s-} + h_s^2 \hat{Y}_{s-} + h_s^3 \hat{Z}_{s-})\tilde{\mu}(dsde), \\ d\hat{Y}_s &= -(g_s^1 \hat{X}_s + g_s^2 \hat{Y}_s + g_s^3 \hat{Z}_s + g_s^4 \int_E \hat{K}_s(e)l(e)\lambda(de))ds + \hat{Z}_s dB_s + \int_E \hat{K}_s(e)\tilde{\mu}(dsde), \\ \hat{X}_t &= 0, \\ \hat{Y}_{t+\delta} &= \bar{\Phi} \hat{X}_{t+\delta} + \Phi_1(X_{t+\delta}^2) - \Phi_2(X_{t+\delta}^2), \end{cases} \quad (3.23)$$

where

$$\bar{\Phi} = \begin{cases} \frac{\Phi_1(X_{t+\delta}^1) - \Phi_1(X_{t+\delta}^2)}{X_{t+\delta}^1 - X_{t+\delta}^2}, & \hat{X}_{t+\delta} \neq 0, \\ 0, & \text{otherwise;} \end{cases} \quad l_s^1 = \begin{cases} \frac{l(s, X_s^1, Y_s^1, Z_s^1) - l(s, X_s^2, Y_s^1, Z_s^1)}{X_s^1 - X_s^2}, & \hat{X}_s \neq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$l_s^2 = \begin{cases} \frac{l(s, X_s^2, Y_s^1, Z_s^1) - l(s, X_s^2, Y_s^2, Z_s^1)}{Y_s^1 - Y_s^2}, & \hat{Y}_s \neq 0, \\ 0, & \text{otherwise;} \end{cases} \quad l_s^3 = \begin{cases} \frac{l(s, X_s^2, Y_s^2, Z_s^1) - l(s, X_s^2, Y_s^2, Z_s^2)}{Z_s^1 - Z_s^2}, & \hat{Z}_s \neq 0, \\ 0, & \text{otherwise;} \end{cases}$$

where $l(\cdot) = b(\cdot)$, $\sigma(\cdot)$, $h(\cdot, e)$, respectively, when $l = h$, in the above representation, X_s^1 , X_s^2 , Y_s^1 , Y_s^2 become X_{s-}^1 , X_{s-}^2 , Y_{s-}^1 , Y_{s-}^2 , respectively, and

$$g_s^1 = \begin{cases} \frac{g(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - g(s, X_s^2, Y_s^1, Z_s^1, K_s^1)}{X_s^1 - X_s^2}, & \hat{X}_s \neq 0, \\ 0, & \text{otherwise;} \end{cases} \quad g_s^2 = \begin{cases} \frac{g(s, X_s^2, Y_s^1, Z_s^1, K_s^1) - g(s, X_s^2, Y_s^2, Z_s^1, K_s^1)}{Y_s^1 - Y_s^2}, & \hat{Y}_s \neq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$g_s^3 = \begin{cases} \frac{g(s, X_s^2, Y_s^2, Z_s^1, K_s^1) - g(s, X_s^2, Y_s^2, Z_s^2, K_s^1)}{Z_s^1 - Z_s^2}, & \hat{Z}_s \neq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$g_s^4 = \begin{cases} \frac{g(s, X_s^2, Y_s^2, Z_s^2, K_s^1) - g(s, X_s^2, Y_s^2, Z_s^2, K_s^2)}{\int_E K_s^1(e)l(e)\lambda(de) - \int_E K_s^2(e)l(e)\lambda(de)}, & \int_E \hat{K}_s(e)l(e)\lambda(de) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It's easy to check that (3.23) satisfies **(H2.1)**, **(H3.1)**, **(H3.3)**. Therefore, from Proposition 3.2, there exists a constant $0 < \delta_1 \leq \delta_0$, such that for every $0 \leq \delta \leq \delta_1$, (3.23) has a unique solution $(\hat{X}, \hat{Y}, \hat{Z}, \hat{K})$ on $[t, t+\delta]$. Now we want to prove $\hat{Y}_t \geq 0$. For this, we introduce the dual FBSDE with jumps

$$\begin{cases} dP_s &= (g_s^2 P_s - b_s^2 Q_s - \sigma_s^2 M_s - h_s^2 N_s)ds + (g_s^3 P_s - b_s^3 Q_s - \sigma_s^3 M_s - h_s^3 N_s)dB_s + \int_E g_s^4 P_{s-} l(e)\tilde{\mu}(dsde), \\ dQ_s &= (g_s^1 P_s - b_s^1 Q_s - \sigma_s^1 M_s - h_s^1 N_s)ds + M_s dB_s + \int_E N_{s-}(e)\tilde{\mu}(dsde), \\ P_t &= 1, \\ Q_{t+\delta} &= -\bar{\Phi} P_{t+\delta}. \end{cases} \quad (3.24)$$

Notice that also (3.24) satisfies **(H2.1)**, **(H3.1)**, **(H3.3)**. Consequently, due to Theorem 3.2, there exists a constant $0 < \delta_2 \leq \delta_1$, such that for every $0 \leq \delta \leq \delta_2$, (3.24) has a unique solution (P, Q, M, N) on $[t, t+\delta]$. Applying Itô's formula to $\hat{X}_s Q_s + \hat{Y}_s P_s$, we deduce from the equations (3.23) and (3.24),

$$\hat{Y}_t = E[(\Phi_1(X_{t+\delta}^2) - \Phi_2(X_{t+\delta}^2))P_{t+\delta} | \mathcal{F}_t].$$

Since $\Phi_1(X_{t+\delta}^2) \geq \Phi_2(X_{t+\delta}^2)$, P-a.s., if we can prove $P_{t+\delta} \geq 0$, P-a.s., then we get $\hat{Y}_t \geq 0$, P-a.s. For this we define the following stopping time: $\tau = \inf\{s > t : P_s \leq 0\} \wedge (t + \delta)$. So, $\tau \leq t + \delta$, a.s. and $P_{\tau-} \geq 0$. In the

first equation of (3.24), the jumps of P_t are only produced by the random measure μ , from **(H2.1)**-(ii) and $l \geq 0$ on E ,

$$\Delta P_\tau \geq 0, \quad P_\tau = P_{\tau-} + \Delta P_\tau \geq 0.$$

Therefore, $P_\tau = 0$, when $\tau < t + \delta$, and $P_\tau \geq 0$, when $\tau = t + \delta$. Consider the following FBSDE on $[\tau, t + \delta]$:

$$\begin{cases} d\tilde{P}_s &= (g_s^2 \tilde{P}_s - b_s^2 \tilde{Q}_s - \sigma_s^2 \tilde{M}_s - h_s^2 \tilde{N}_s)ds + (g_s^3 \tilde{P}_s - b_s^3 \tilde{Q}_s - \sigma_s^3 \tilde{M}_s - h_s^3 \tilde{N}_s)dB_s + \int_E g_s^4 \tilde{P}_s l(e) \tilde{\mu}(dsde), \\ d\tilde{Q}_s &= (g_s^1 \tilde{P}_s - b_s^1 \tilde{Q}_s - \sigma_s^1 \tilde{M}_s - h_s^1 \tilde{N}_s)ds - \tilde{M}_s dB_s - \int_E \tilde{N}_s \tilde{\mu}(dsde), \\ \tilde{P}_\tau &= 0, \\ \tilde{M}_{t+\delta} &= -\tilde{\Phi} \tilde{P}_{t+\delta}. \end{cases} \quad (3.25)$$

Due to Theorem 3.2 there exists $0 < \delta_3 \leq \delta_2$ such that for every $0 \leq \delta \leq \delta_3$, (3.25) has a unique solution $(\tilde{P}, \tilde{Q}, \tilde{M}_s, \tilde{N}_s)$ on $[\tau, t + \delta]$. Clearly, $(\tilde{P}_s, \tilde{Q}_s, \tilde{M}_s, \tilde{N}_s) \equiv (0, 0, 0, 0)$ is the unique solution of (3.25). Let

$$\begin{aligned} \bar{P}_s &= I_{[t, \tau]}(s)P_s + I_{(\tau, t+\delta]}(s)\tilde{P}_s, & \bar{Q}_s &= I_{[t, \tau]}(s)Q_s + I_{(\tau, t+\delta]}(s)\tilde{Q}_s, \\ \bar{M}_s &= I_{[t, \tau]}(s)M_s + I_{(\tau, t+\delta]}(s)\tilde{M}_s, & \bar{N}_s &= I_{[t, \tau]}(s)N_s + I_{(\tau, t+\delta]}(s)\tilde{N}_s, \quad s \in [t, t + \delta]. \end{aligned}$$

Considering that $P_\tau = 0$ on $\{\tau < t + \delta\}$, it's easy to show that $(\bar{P}, \bar{Q}, \bar{M}, \bar{N})$ is a solution of FBSDE (3.24). Therefore, from the uniqueness of solution of FBSDE (3.24) on $[t, t + \delta]$, where $0 \leq \delta \leq \delta_3$, we have $\bar{P}_t = P_t = 1 > 0$. Furthermore, from the definition of τ we have $\bar{P}_{t+\delta} \geq 0$, P-a.s., that is, $P_{t+\delta} \geq 0$, P-a.s. Therefore, we have $Y_t^1 \geq Y_t^2$, P-a.s. □

In order to derive some regularity results, we need the following condition:

(H3.4) For any $t \in [0, T]$, for any $(x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$, P-a.s., $|h(t, x, y, z, e)| \leq \rho(e)(1 + |x| + |y|)$, where $\rho(e) = C(1 \wedge |e|)$.

Theorem 3.4. *Let Φ be deterministic, and suppose the assumptions **(H2.1)**, **(H3.1)**, **(H3.3)**, **(H3.4)** hold true. Then, for every $p \geq 2$, there exists a sufficiently small constant $\tilde{\delta} > 0$, only depending on the Lipschitz constants K and L_σ , $L_h(\cdot)$, and some constant $\tilde{C}_{p,K}$, only depending on p , the Lipschitz constants K , L_σ , $L_h(\cdot)$ and the linear growth constant L , such that for every $0 \leq \delta \leq \tilde{\delta}$ and $\zeta \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,*

$$\begin{aligned} \text{(i)} \quad & E\left[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta}|^p + \sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p + \left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds\right)^{\frac{p}{2}} \right. \\ & \left. + \left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds\right)^{\frac{p}{2}} | \mathcal{F}_t \right] \leq \tilde{C}_{p,K}(1 + |\zeta|^p), \quad P\text{-a.s.}; \\ \text{(ii)} \quad & E\left[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta} - \zeta|^p | \mathcal{F}_t\right] \leq \tilde{C}_{p,K}\delta(1 + |\zeta|^p), \quad P\text{-a.s.}, \\ \text{(iii)} \quad & E\left[\left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds\right)^{\frac{p}{2}} + \left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds\right)^{\frac{p}{2}} | \mathcal{F}_t\right] \leq \tilde{C}_{p,K}\delta^{\frac{p}{2}}(1 + |\zeta|^p), \quad P\text{-a.s.}, \end{aligned}$$

where $(X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}, K_s^{t,\zeta})_{s \in [t, t+\delta]}$ is the solution of FBSDE (3.22) associated with $(b, \sigma, g, \zeta, \Phi)$ and with the time horizon $t + \delta$.

Proof. Without loss of generality, we restrict ourselves to the proof for $p = 2k$, $k \in \mathbb{Z}^+$.

Due to Theorem 3.2, there exists a constant $\delta_0 > 0$ depending on K , L_σ , \tilde{C}_h , such that for every $0 \leq \delta \leq \delta_0$, (3.22) has a unique solution on $[t, t + \delta]$, i.e.,

$$Y_s^{t,\zeta} = \Phi(X_{t+\delta}^{t,\zeta}) + \int_s^{t+\delta} g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta})dr - \int_s^{t+\delta} Z_r^{t,\zeta} dB_r - \int_s^{t+\delta} \int_E K_r^{t,\zeta}(e) \tilde{\mu}(drde). \quad (3.26)$$

Set $\tilde{Y}_s^{t,\zeta} = Y_s^{t,\zeta} - \Phi(\zeta)$. For any $\beta \geq 0$, by applying Itô's formula to $e^{\beta s} |\tilde{Y}_s^{t,\zeta}|^2$, taking β large enough, using BSDE standard methods, and by considering that $|g(r, \zeta, \Phi(\zeta), 0, 0)| \leq C(1 + |\zeta|)$, we get

$$\begin{aligned} & |\tilde{Y}_s^{t,\zeta}|^2 + E\left[\int_s^{t+\delta} (|\tilde{Y}_r^{t,\zeta}|^2 + |Z_r^{t,\zeta}|^2 + \int_E |K_r^{t,\zeta}(e)|^2 \lambda(de))dr | \mathcal{F}_s\right] \\ & \leq CE\left[\sup_{s \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^2 | \mathcal{F}_s\right] + C(t + \delta - s)(1 + |\zeta|^2), \quad P\text{-a.s.}, \end{aligned} \quad (3.27)$$

where C only depends on K and L . Therefore, from (3.26) and (3.27) and Burkholder-Davis-Gundy inequality,

$$E\left[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^2 | \mathcal{F}_t\right] \leq CE\left[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^2 | \mathcal{F}_t\right] + C\delta(1 + |\zeta|^2), \text{ P-a.s.} \quad (3.28)$$

On the other hand, from (3.27)

$$|\tilde{Y}_s^{t,\zeta}|^2 \leq CE\left[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^2 | \mathcal{F}_s\right] + C\delta(1 + |\zeta|^2), \text{ P-a.s., } t \leq s \leq t + \delta. \quad (3.29)$$

When $p > 2$, we define $\eta = \sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^2 \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R}^n)$. Then $M_s := E[\eta | \mathcal{F}_s], s \in [t, t + \delta]$, is a martingale, and from Doob's martingale inequality we have

$$E\left[\sup_{t \leq s \leq t+\delta} |M_s|^{\frac{p}{2}} | \mathcal{F}_t\right] \leq C_p E[|M_{t+\delta}|^{\frac{p}{2}} | \mathcal{F}_t] \leq C_p E[\eta^{\frac{p}{2}} | \mathcal{F}_t] = C_p E\left[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p | \mathcal{F}_t\right], \text{ P-a.s.} \quad (3.30)$$

Therefore, from (3.29) and (3.30)

$$E\left[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^p | \mathcal{F}_t\right] \leq C_p E\left[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p | \mathcal{F}_t\right] + C_p \delta^{\frac{p}{2}}(1 + |\zeta|^p), \text{ P-a.s.} \quad (3.31)$$

Now we consider

$$Y_s^{t,\zeta} - \Phi(\zeta) = \Phi(X_{t+\delta}^{t,\zeta}) - \Phi(\zeta) + \int_s^{t+\delta} g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta}) dr - \int_s^{t+\delta} Z_r^{t,\zeta} dB_r - \int_s^{t+\delta} \int_E K_r^{t,\zeta}(e) \tilde{\mu}(dsde).$$

From Burkholder-Davis-Gundy inequality and (3.11), (3.31),

$$\begin{aligned} & E\left[\left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds\right)^{\frac{p}{2}} | \mathcal{F}_t\right] + E\left[\left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds\right)^{\frac{p}{2}} | \mathcal{F}_t\right] \\ & \leq E\left[\left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds\right)^{\frac{p}{2}} | \mathcal{F}_t\right] + C_p E\left[\left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \mu(dsde)\right)^{\frac{p}{2}} | \mathcal{F}_t\right] \\ & \leq C_p E\left[\sup_{t \leq s \leq t+\delta} \left|\int_t^s Z_r^{t,\zeta} dB_r + \int_t^s \int_E K_r^{t,\zeta}(e) \tilde{\mu}(dsde)\right|^p | \mathcal{F}_t\right] \\ & \leq C_p E\left[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^p + \left(\int_t^{t+\delta} |g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}, K_s^{t,\zeta})| ds\right)^p | \mathcal{F}_t\right] \\ & = C_p E\left[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^p | \mathcal{F}_t\right] + C_p E\left[\left(\int_t^{t+\delta} |g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}, K_s^{t,\zeta}) - g(s, \zeta, \Phi(\zeta), 0, 0)| \right. \right. \\ & \quad \left. \left. + g(s, \zeta, \Phi(\zeta), 0, 0)| ds\right)^p | \mathcal{F}_t\right] \\ & \leq (C_p + C_p \delta^p) E\left[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta} - \zeta|^p | \mathcal{F}_t\right] + C_p \delta^{\frac{p}{2}}(1 + |\zeta|^p) + C_p \delta^{\frac{p}{2}} E\left[\left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds\right)^{\frac{p}{2}} | \mathcal{F}_t\right] \\ & \quad + C_p \delta^{\frac{p}{2}} E\left[\left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds\right)^{\frac{p}{2}} | \mathcal{F}_t\right]. \end{aligned} \quad (3.32)$$

By choosing $0 < \delta_1 \leq \delta_0$ such that $1 - C_p \delta_1^{\frac{p}{2}} > 0$, we get, for any $0 \leq \delta \leq \delta_1$, P-a.s.,

$$\begin{aligned} & E\left[\left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds\right)^{\frac{p}{2}} | \mathcal{F}_t\right] + E\left[\left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds\right)^{\frac{p}{2}} | \mathcal{F}_t\right] \\ & \leq (C_p + C_p \delta^p) E\left[\sup_{t \leq s \leq T} |X_s^{t,\zeta} - \zeta|^p | \mathcal{F}_t\right] + C_p \delta^{\frac{p}{2}}(1 + |\zeta|^p). \end{aligned} \quad (3.33)$$

Therefore, from the second line and the latter estimate of (3.32) we know

$$\begin{aligned} & E\left[\left(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds\right)^{\frac{p}{2}} | \mathcal{F}_t\right] + E\left[\left(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \mu(dsde)\right)^{\frac{p}{2}} | \mathcal{F}_t\right] \\ & \leq (C_p + C_p \delta^p) E\left[\sup_{t \leq s \leq T} |X_s^{t,\zeta} - \zeta|^p | \mathcal{F}_t\right] + C_p \delta^{\frac{p}{2}}(1 + |\zeta|^p), \text{ P-a.s.} \end{aligned} \quad (3.34)$$

Similarly, equation (3.22) and the estimates (3.31), (3.33), (3.34) yield

$$\begin{aligned} & E\left[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p | \mathcal{F}_t\right] \\ & \leq C_p E\left[\left(\int_t^{t+\delta} b(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}) dr\right)^p | \mathcal{F}_t\right] + C_p E\left[\left(\int_t^{t+\delta} |\sigma(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})|^2 dr\right)^{\frac{p}{2}} | \mathcal{F}_t\right] \\ & \quad + C_p E\left[\left(\int_t^{t+\delta} \int_E |h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_r^{t,\zeta}, e)|^2 \mu(drde)\right)^{\frac{p}{2}} | \mathcal{F}_t\right] \\ & \leq C_p E\left[\left(\int_t^{t+\delta} (1 + |X_r^{t,\zeta} - \zeta| + |\zeta| + |Y_r^{t,\zeta} - \Phi(\zeta)| + |Z_r^{t,\zeta}|) dr\right)^p | \mathcal{F}_t\right] \\ & \quad + C_p E\left[\left(\int_t^{t+\delta} (1 + |X_r^{t,\zeta} - \zeta|^2 + |\zeta|^2 + |Y_r^{t,\zeta} - \Phi(\zeta)|^2 + L_\sigma^2 |Z_r^{t,\zeta}|^2) dr\right)^{\frac{p}{2}} | \mathcal{F}_t\right] \\ & \quad + C_p E\left[\left(\int_t^{t+\delta} \int_E |h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_r^{t,\zeta}, e)|^2 \mu(drde)\right)^{\frac{p}{2}} | \mathcal{F}_t\right], \end{aligned} \quad (3.35)$$

where

$$\begin{aligned}
& E[(\int_t^{t+\delta} \int_E |h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_{r-}^{t,\zeta}, e)|^2 \mu(drde))^\frac{p}{2} | \mathcal{F}_t] \\
& \leq E[(\int_t^{t+\delta} \int_E C(1 \wedge |e|^2)(1 + |X_{r-}^{t,\zeta}| + |Y_{r-}^{t,\zeta}|)^2 \mu(drde))^\frac{p}{2} | \mathcal{F}_t] \\
& \leq E[(\int_t^{t+\delta} \int_E C_p(1 \wedge |e|^2)(1 + |X_{r-}^{t,\zeta}|^2 + |Y_{r-}^{t,\zeta}|^2) \mu(drde))^\frac{p}{2} | \mathcal{F}_t] \\
& \leq E[(\int_t^{t+\delta} \int_E C_p(1 \wedge |e|^2)(1 + |X_{r-}^{t,\zeta} - \zeta|^2 + |Y_{r-}^{t,\zeta} - \Phi(\zeta)|^2 + |\zeta|^2) \mu(drde))^\frac{p}{2} | \mathcal{F}_t] \\
& \leq E[(\int_t^{t+\delta} \int_E C_p(1 \wedge |e|^2) \mu(drde))^\frac{p}{2} | \mathcal{F}_t] (1 + |\zeta|^p) + C_p E[(\int_t^{t+\delta} \int_E (1 \wedge |e|^2) |X_{r-}^{t,\zeta} - \zeta|^2 \mu(drde))^\frac{p}{2} | \mathcal{F}_t] \\
& \quad + C_p E[(\int_t^{t+\delta} \int_E (1 \wedge |e|^2) |\tilde{Y}_{r-}^{t,\zeta}|^2 \mu(drde))^\frac{p}{2} | \mathcal{F}_t] \\
& \leq C_p \delta (1 + |\zeta|^p) + C_p \delta E[\sup_{t \leq s \leq t+\delta} |X_{r-}^{t,\zeta} - \zeta|^p | \mathcal{F}_t] + C_p \delta E[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^p | \mathcal{F}_t],
\end{aligned} \tag{3.36}$$

where we have used that

$$E[(\int_t^{t+\delta} \int_E (1 \wedge |e|^2) |\tilde{Y}_{r-}^{t,\zeta}|^2 \mu(drde))^\frac{p}{2} | \mathcal{F}_t] \leq C_p \delta E[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^p | \mathcal{F}_t]. \tag{3.37}$$

Indeed, we denote $\gamma_s(e) := (1 \wedge |e|^2) |\tilde{Y}_s^{t,\zeta}|^2$, $\bar{\gamma}_s(e) := |\tilde{Y}_s^{t,\zeta}|^2$, $A_r := \int_t^{t+\delta} \int_E \gamma_s(e) \mu(dsde)$. Similarly to (3.15) in the proof of Proposition 3.2 we can prove that (3.37).

In the same way, we have

$$E[|\int_t^r \int_E |X_{s-}^{t,\zeta} - \zeta|^2 (1 \wedge |e|^2) \mu(dsde)|^\frac{p}{2} | \mathcal{F}_t] \leq C_p(r-t) E[\sup_{t \leq s \leq r} |X_s^{t,\zeta} - \zeta|^p | \mathcal{F}_t],$$

and

$$E[|\int_t^r \int_E (1 \wedge |e|^2) \mu(dsde)|^\frac{p}{2} | \mathcal{F}_t] \leq C_p(r-t).$$

From (3.35), we have

$$\begin{aligned}
E[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p | \mathcal{F}_t] & \leq C_p(1 + |\zeta|^p) \delta + C_p(\delta^\frac{p}{2} + L_\sigma^p) E[(\int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr)^\frac{p}{2} | \mathcal{F}_t] \\
& \quad + C_p \delta E[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p | \mathcal{F}_t] + C_p \delta E[\sup_{t \leq r \leq t+\delta} |Y_r^{t,\zeta} - \Phi(\zeta)|^p | \mathcal{F}_t].
\end{aligned} \tag{3.38}$$

From (3.31), (3.34) and (3.38), we get

$$\begin{aligned}
& E[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p | \mathcal{F}_t] \\
& \leq C_p \delta (1 + |\zeta|^p) + C_p(\delta + \delta^\frac{p}{2} + \delta^\frac{3p}{2} + L_\sigma^p + L_\sigma^p \delta^p) E[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p | \mathcal{F}_t], \text{ P-a.s., } 0 \leq \delta \leq \delta_0.
\end{aligned} \tag{3.39}$$

Due to L_σ is sufficiently small, we choose L_σ satisfying $C_p L_\sigma^p < 1$. Then there exists a constant $0 < \delta_2 \leq \delta_1$ such that $1 - C_p(\delta_2 + \delta_2^\frac{p}{2} + \delta_2^\frac{3p}{2} + L_\sigma^p + L_\sigma^p \delta_2^p) > 0$, therefore we get for any $0 \leq \delta \leq \delta_2$, P-a.s.

$$E[\sup_{t \leq r \leq t+\delta} |X_r^{t,\zeta} - \zeta|^p | \mathcal{F}_t] \leq C_p \delta (1 + |\zeta|^p). \tag{3.40}$$

Furthermore, from (3.31), (3.33), (3.34), and (3.40),

$$E[\sup_{t \leq s \leq t+\delta} |X_s^{t,\zeta}|^p + \sup_{t \leq s \leq t+\delta} |Y_s^{t,\zeta}|^p + (\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^\frac{p}{2} + (\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds)^\frac{p}{2} | \mathcal{F}_t] \leq \tilde{C}_{p,K}(1 + |\zeta|^p), \text{ P-a.s.,} \tag{3.41}$$

and

$$E[(\int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \mu(dsde))^\frac{p}{2} | \mathcal{F}_t] \leq \tilde{C}_{p,K}(1 + |\zeta|^p), \text{ P-a.s.}$$

Now we prove (iii). For convenience, we denote

$$\begin{aligned}
\theta_s^{t,\zeta} &:= \int_t^s \int_E h(r, X_{r-}^{t,\zeta}, Y_{r-}^{t,\zeta}, Z_{r-}^{t,\zeta}, e) \mu(drde), \quad \eta_s^{t,\zeta}(e) := \Phi(\zeta + \theta_s^{t,\zeta} + h(s, \Pi_s^{t,\zeta}, e)) - \Phi(\zeta + \theta_{s-}^{t,\zeta}), \\
\tilde{Y}_s^{t,\zeta} &:= \int_t^s \int_E \eta_r^{t,\zeta}(e) \tilde{\mu}(drde), \quad \tilde{X}_s^{t,\zeta} := X_s^{t,\zeta} - \theta_s^{t,\zeta}, \quad \tilde{Y}_s^{t,\zeta} := Y_s^{t,\zeta} - \tilde{Y}_s^{t,\zeta} - \Phi(\zeta), \\
\tilde{K}_s^{t,\zeta}(e) &:= K_s^{t,\zeta}(e) - \eta_s^{t,\zeta}(e), \quad \Pi_{s-}^{t,\zeta} := (X_{s-}^{t,\zeta}, Y_{s-}^{t,\zeta}, Z_{s-}^{t,\zeta}).
\end{aligned}$$

We know

$$|\eta_s^{t,\zeta}(e)| \leq C|h(s, \Pi_{s-}^{t,\zeta}, e)| \leq C(1 \wedge |e|^2)(1 + |X_{s-}^{t,\zeta}| + |Y_{s-}^{t,\zeta}|). \quad (3.42)$$

And it is easy to check

$$\begin{aligned} \Phi(\zeta + \theta_{t+\delta}^{t,\zeta}) - \Phi(\zeta) &= \sum_{t < s \leq t+\delta} (\Phi(\zeta + \theta_s^{t,\zeta}) - \Phi(\zeta + \theta_{s-}^{t,\zeta})) \\ &= \sum_{t < s \leq t+\delta} \int_E [\Phi(\zeta + \theta_{s-}^{t,\zeta} + h(s, \Pi_{s-}^{t,\zeta}, e)) - \Phi(\zeta + \theta_{s-}^{t,\zeta})] \mu(\{s\}, de) \\ &= \int_t^{t+\delta} \int_E [\Phi(\zeta + \theta_{s-}^{t,\zeta} + h(s, \Pi_{s-}^{t,\zeta}, e)) - \Phi(\zeta + \theta_{s-}^{t,\zeta})] \mu(dsde) = \int_t^{t+\delta} \int_E \eta_s^{t,\zeta}(e) \mu(dsde). \end{aligned}$$

Therefore,

$$\Phi(\zeta + \theta_{t+\delta}^{t,\zeta}) - \Phi(\zeta) - \int_t^{t+\delta} \int_E \eta_s^{t,\zeta}(e) \lambda(de) ds = \int_t^{t+\delta} \int_E \eta_s^{t,\zeta}(e) \tilde{\mu}(dsde).$$

Then,

$$\hat{Y}_s^{t,\zeta} = \Phi(\zeta + \theta_{t+\delta}^{t,\zeta}) - \Phi(\zeta) - \int_t^{t+\delta} \int_E \eta_s^{t,\zeta}(e) \lambda(de) ds - \int_s^{t+\delta} \int_E \eta_s^{t,\zeta}(e) \tilde{\mu}(dsde).$$

From equation (3.22), we have

$$\begin{cases} d\tilde{X}_s^{t,\zeta} &= b(s, \tilde{X}_s^{t,\zeta} + \theta_s^{t,\zeta}, \tilde{Y}_s^{t,\zeta} + \hat{Y}_s^{t,\zeta} + \Phi(\zeta), Z_s^{t,\zeta}) ds + \sigma(s, \tilde{X}_s^{t,\zeta} + \theta_s^{t,\zeta}, \tilde{Y}_s^{t,\zeta} + \hat{Y}_s^{t,\zeta} + \Phi(\zeta), Z_s^{t,\zeta}) dB_s \\ &\quad + \int_E h(s, \tilde{X}_s^{t,\zeta} + \theta_s^{t,\zeta}, \tilde{Y}_s^{t,\zeta} + \hat{Y}_s^{t,\zeta} + \Phi(\zeta), Z_s^{t,\zeta}, e) \lambda(de) ds, \quad s \in [t, t+\delta], \\ d\tilde{Y}_s^{t,\zeta} &= -g(s, \tilde{X}_s^{t,\zeta} + \theta_s^{t,\zeta}, \tilde{Y}_s^{t,\zeta} + \hat{Y}_s^{t,\zeta} + \Phi(\zeta), Z_s^{t,\zeta}, \tilde{K}_s^{t,\zeta} + \eta_s^{t,\zeta}) ds + Z_s^{t,\zeta} dB_s + \int_E \tilde{K}_s^{t,\zeta}(e) \tilde{\mu}(dsde), \\ \tilde{X}_t^{t,\zeta} &= \zeta, \\ \tilde{Y}_{t+\delta}^{t,\zeta} &= \Phi(\tilde{X}_{t+\delta}^{t,\zeta} + \theta_{t+\delta}^{t,\zeta}) - \Phi(\zeta + \theta_{t+\delta}^{t,\zeta}) + \int_t^{t+\delta} \int_E \eta_s^{t,\zeta}(e) \lambda(de) ds. \end{cases} \quad (3.43)$$

For $(X^{t,\zeta}, Y^{t,\zeta}, Z^{t,\zeta}, K^{t,\zeta})$, (3.41) holds true, for any $\delta \in [0, \tilde{\delta}]$. For the backward part of equation (3.43),

$$|\tilde{Y}_s^{t,\zeta}| \leq E[|\tilde{Y}_{t+\delta}^{t,\zeta}| + \int_t^{t+\delta} |g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta})| dr \mid \mathcal{F}_s], \quad s \in [t, t+\delta].$$

From (3.41) and Doob's martingale inequality for $p > 2$,

$$\begin{aligned} E[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^p \mid \mathcal{F}_t] &\leq C_p E[|\tilde{Y}_{t+\delta}^{t,\zeta}|^p \mid \mathcal{F}_t] + C_p \delta^{\frac{p}{2}} E[(\int_t^{t+\delta} |g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta})|^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t] \\ &\leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p) + C_p E[|\tilde{Y}_{t+\delta}^{t,\zeta}|^p \mid \mathcal{F}_t]. \end{aligned} \quad (3.44)$$

We need to estimate $|\tilde{Y}_{t+\delta}^{t,\zeta}|$, and notice

$$|\tilde{Y}_{t+\delta}^{t,\zeta}| \leq C|\tilde{X}_{t+\delta}^{t,\zeta} - \zeta| + \int_t^{t+\delta} \int_E |\eta_s^{t,\zeta}(e)| \lambda(de) ds. \quad (3.45)$$

From (3.41) and (3.42), we get

$$E[(\int_t^{t+\delta} \int_E |\eta_s^{t,\zeta}(e)| \lambda(de) ds)^p \mid \mathcal{F}_t] \leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p). \quad (3.46)$$

On the other hand, from (3.41), we have

$$\begin{aligned} E[\int_t^s |b(s, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})| dr]^p \mid \mathcal{F}_t &\leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p), \\ E[\int_t^s |\sigma(s, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})| dB_r]^p \mid \mathcal{F}_t &\leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p) + C_p L_\sigma^p E[(\int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t], \\ E[(\int_t^{t+\delta} \int_E |h(s, X_{s-}^{t,\zeta}, Y_{s-}^{t,\zeta}, Z_{s-}^{t,\zeta}, e)| \lambda(de) ds)^p \mid \mathcal{F}_t] &\leq C_p E[(\int_E (1 \wedge |e|^2) \lambda(de))^p (\int_t^{t+\delta} (1 + |X_s^{t,\zeta}| + |Y_s^{t,\zeta}|) ds)^p \mid \mathcal{F}_t] \leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p). \end{aligned} \quad (3.47)$$

From (3.43) and the above estimates (3.47), we know

$$E[\sup_{t \leq s \leq t+\delta} |\tilde{X}_{t+\delta}^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t] \leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p) + C_p L_\sigma^p E[(\int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t]. \quad (3.48)$$

From (3.44), (3.45), (3.46) and (3.48),

$$E\left[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^p \mid \mathcal{F}_t\right] \leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p) + C_p L_\sigma^p E\left[\left(\int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right]. \quad (3.49)$$

From Burkholder-Davis-Gundy inequality and (3.41), we have

$$\begin{aligned} & E\left[\left(\int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] + E\left[\left(\int_t^{t+\delta} \int_E |\tilde{K}_r^{t,\zeta}(e)|^2 \mu(drde)\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] \\ & \leq C_p E\left[\sup_{t \leq s \leq t+\delta} \left|\int_t^s Z_r^{t,\zeta} dB_r + \int_t^s \int_E \tilde{K}_r^{t,\zeta}(e) \tilde{\mu}(drde)\right|^p \mid \mathcal{F}_t\right] \\ & \leq C_p E\left[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^p \mid \mathcal{F}_t\right] + C_p E\left[\left(\int_t^{t+\delta} |g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta})| dr\right)^p \mid \mathcal{F}_t\right] \\ & \leq C_p E\left[\sup_{t \leq s \leq t+\delta} |\tilde{Y}_s^{t,\zeta}|^p \mid \mathcal{F}_t\right] + C_p \delta^{\frac{p}{2}} E\left[\left(\int_t^{t+\delta} (1 + |X_r^{t,\zeta}|^2 + |Y_r^{t,\zeta}|^2 + |Z_r^{t,\zeta}|^2) dr\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] \\ & \quad + C_p E\left[\left(\int_t^{t+\delta} \int_E |K_r^{t,\zeta}(e)| (1 \wedge |e|) \lambda(de) dr\right)^p \mid \mathcal{F}_t\right] \\ & \leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p) + C_p L_\sigma^p E\left[\left(\int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right]. \end{aligned} \quad (3.50)$$

As L_σ is sufficiently small, for $C_p L_\sigma^p < 1$, we have

$$E\left[\left(\int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] + E\left[\left(\int_t^{t+\delta} \int_E |\tilde{K}_r^{t,\zeta}(e)|^2 \mu(drde)\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] \leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p). \quad (3.51)$$

From (3.11), (3.46), (3.51) and $K_s^{t,\zeta}(e) = \tilde{K}_s^{t,\zeta}(e) + \eta_s^{t,\zeta}(e)$, we get

$$E\left[\left(\int_t^{t+\delta} \int_E |K_r^{t,\zeta}(e)|^2 \lambda(de) dr\right)^{\frac{p}{2}} \mid \mathcal{F}_t\right] \leq C_p \delta^{\frac{p}{2}} (1 + |\zeta|^p).$$

Therefore, the estimate (iii) is derived. \square

Remark 3.6. If the initial state $\zeta = x \in \mathbb{R}^n$ is given, the terminal condition Φ becomes $\Phi(x)$, that is, FBSDE (3.22) becomes the following

$$\begin{cases} dX_s &= b(s, X_s, Y_s, Z_s) ds + \sigma(s, X_s, Y_s, Z_s) dB_s + \int_E h(s, X_{s-}, Y_{s-}, Z_s, e) \tilde{\mu}(dsde), \\ dY_s &= -g(s, X_s, Y_s, Z_s, K_s) ds + Z_s dB_s + \int_E K_s(e) \tilde{\mu}(dsde), \quad s \in [t, t+\delta], \\ X_t &= x, \\ Y_{t+\delta} &= \Phi(x), \end{cases} \quad (3.52)$$

then Theorem 3.4 still holds.

Indeed, from Lemma 2.2, FBSDE (3.52) has a unique solution (X, Y, Z, K) . We consider the following FBSDE:

$$\begin{cases} d\hat{X}_s &= b(s, \hat{X}_s, \hat{Y}_s + \Phi(x), \hat{Z}_s) ds + \sigma(s, \hat{X}_s, \hat{Y}_s + \Phi(x), \hat{Z}_s) dB_s + \int_E h(s, \hat{X}_{s-}, \hat{Y}_{s-} + \Phi(x), \hat{Z}_s, e) \tilde{\mu}(dsde), \\ d\hat{Y}_s &= -g(s, \hat{X}_s, \hat{Y}_s + \Phi(x), \hat{Z}_s, \hat{K}_s) ds + \hat{Z}_s dB_s + \int_E \hat{K}_s(e) \tilde{\mu}(dsde), \quad s \in [t, t+\delta], \\ \hat{X}_t &= x, \\ \hat{Y}_{t+\delta} &= 0. \end{cases} \quad (3.53)$$

From Lemma 2.2, we know $(X, Y, Z, K) = (\hat{X}, \hat{Y} + \Phi(x), \hat{Z}, \hat{K})$. For $(\hat{X}, \hat{Y}, \hat{Z}, \hat{K})$, Theorem 3.4 holds, which means those estimates in Theorem 3.4 still holds for (X, Y, Z, K) .

Proposition 3.3. Suppose that $(b_i, \sigma_i, g_i, \Phi_i)$, $i = 1, 2$, all satisfy the assumptions **(H2.1)**, **(H3.1)**, **(H3.3)**. Then from Theorem 3.3 there exists a constant $0 < \delta_0$, only depending on the Lipschitz constants K , L_σ and $L_h(\cdot)$, such that for $0 \leq \delta \leq \delta_0$, and the same initial state $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, $(X_s^i, Y_s^i, Z_s^i)_{s \in [t, t+\delta]}$ is the solution of FBSDE (3.1) associated with $(b_i, \sigma_i, g_i, \Phi_i)$ on the time interval $[t, t+\delta]$, $i = 1, 2$. It follows that there exists a constant $\delta_1 > 0$, such that for every $0 \leq \delta \leq \delta_1$,

$$\begin{aligned} & |Y_t^1 - Y_t^2|^2 \\ & \leq CE[|\Phi_1(t+\delta, X_{t+\delta}^1) - \Phi_2(t+\delta, X_{t+\delta}^1)|^2 \mid \mathcal{F}_t] + C\delta E\left[\int_t^{t+\delta} |(b_1 - b_2)(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds \mid \mathcal{F}_t\right] \\ & \quad + CE\left[\int_t^{t+\delta} |(\sigma_1 - \sigma_2)(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds \mid \mathcal{F}_t\right] + C\delta E\left[\int_t^{t+\delta} |(g_1 - g_2)(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds \mid \mathcal{F}_t\right] \\ & \quad + CE\left[\int_t^{t+\delta} \int_E |(h_1 - h_2)(s, X_s^1, Y_s^1, Z_s^1, K_s^1(e), e)|^2 \lambda(de) ds \mid \mathcal{F}_t\right], \quad P\text{-a.s.} \end{aligned}$$

For the proof, it is similar to the proof of Proposition 6.6 in Li, Wei [6].

Remark 3.7. When $(b_1, \sigma_1, h_1, f_1) = (b_2, \sigma_2, h_2, f_2)$ in Proposition 3.3, we have

$$|Y_t^1 - Y_t^2| \leq C(E[|\Phi_1(t + \delta, X_{t+\delta}^1) - \Phi_2(t + \delta, X_{t+\delta}^1)|^2 | \mathcal{F}_t])^{\frac{1}{2}}, \quad P\text{-a.s.}$$

Corollary 3.1. Under the assumptions (H2.1), (H3.1), (H3.3), there exists a constant $0 < \delta_0$, only depending on the Lipschitz constants K , L_σ and $L_h(\cdot)$, such that for every $0 \leq \delta \leq \delta_0$, $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ and $\varepsilon > 0$, if $(X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}, K_s^{t,\zeta})_{s \in [t, t+\delta]}$ is the solution of FBSDE (3.1) associated with $(b, \sigma, f, \zeta, \Phi)$, and $(\bar{X}_s^{t,\zeta}, \bar{Y}_s^{t,\zeta}, \bar{Z}_s^{t,\zeta}, \bar{K}_s^{t,\zeta})_{s \in [t, t+\delta]}$ is that of FBSDE (3.1) associated with $(b, \sigma, f, \zeta, \Phi + \varepsilon)$ on the time interval $[t, t + \delta]$, then we have that

$$|Y_t^{t,\zeta} - \bar{Y}_t^{t,\zeta}| \leq C\varepsilon, \quad P\text{-a.s.}$$

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